

Kernel/Metric Correspondence of Dissipative Systems in Information Theory

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ABSTRACT

We study the kernel/metric correspondence pointed out in our previous work in a dissipative system which is accompanying fractional Brownian motion. We also give some comments on information causality.

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1 Introduction

Referring to Amari's seminal works of information geometry [1, 2], we have seen an analogue of the AdS/CFT correspondence in information theory [3].

Following our previous work [3], we study the aforementioned correspondence for the system which consists of stochastically deviating data. The stochastic fluctuation treated in this study follows fractional Brownian motion. We study the kernel/metric correspondence when accompanying with this fluctuation. We also some comments on causality.

This paper is organized as follows: Section two is a short review of kernel in information geometry. Section three is a description of dissipative system. In section four, we see the relationship between stochastic fluctuation and kernel. In section five, we consider the case of fractional Brownian motion. In section six, we give some comments on causality. Section seven is conclusions.

2 Kernel in information geometry

At first, we give the relationship between metric of data space and kernel. This is represented as [2]

$$g_{ij} = \frac{\partial}{\partial x_i} \frac{\partial}{\partial x'_j} K(x_i, x'_j) \Big|_{x_i=x'_j} \quad (1)$$

Here, g_{ij} is the metric of data space, x is point and K is the kernel.

3 Description of dissipative system

Here, we give a short review of Fick's law [4]. For static state, the flux is proportional to gradient of density. This is described as

$$J = -D \frac{\partial c}{\partial x} \quad (2)$$

Here, J is flux, D is diffusion coefficient, and c is density. This is called as Fick's first law. For non-static state, the following equation holds:

$$\frac{\partial c}{\partial t} = -\text{div} J = D \nabla^2 c \quad (3)$$

This is diffusion equation and is equivalent to the equation of continuity. This is called as Fick's second law.

4 Stochastic fluctuation and kernel

Consider the case of stochastic fluctuation of point, which follows standard Brownian motion:

$$dx = \sigma x dz. \quad (4)$$

Bellman equation for kernel derives the diffusion equation:

$$\frac{\partial K}{\partial t} - \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial x^2} K = 0 \quad (5)$$

The solution of this equation (with the boundary condition $K(\infty) = 0$) is,

$$K = \frac{1}{2\sqrt{\pi\sigma^2 t}} \exp\left(-\frac{x^2}{4\sigma^2 t}\right). \quad (6)$$

This is the derivation of Gaussian kernel [5].

5 The case of fractional Brownian motion

5.1 Fractional Brownian motion and kernel on boundary

Next, we consider the case of fractional Brownian motion,

$$d\hat{x} = \sigma \hat{x} dz^H. \quad (7)$$

Here, H is called as Harst constant and we assume $H \in (\frac{1}{2}, 1)$. Infinitesimal differentiation of kernel derives,

$$d\hat{K} = \sigma \hat{x} \frac{\partial \hat{K}}{\partial \hat{x}} (dz^H - H t^{2\alpha} dt) + \sigma^2 H t^{2\alpha} \hat{x}^2 \frac{\partial^2 \hat{K}}{\partial \hat{x}^2} dt, \quad (8)$$

here, $\alpha = H - \frac{1}{2}$. From this equation, we obtain,

$$\frac{\partial \hat{K}}{\partial t} - \sigma^2 H t^{2\alpha} \hat{x}^2 \frac{\partial^2 \hat{K}}{\partial \hat{x}^2} = 0 \quad (9)$$

Substituting $\tau = t^{2H}$ and setting

$$\hat{y} \equiv \frac{1}{\sigma} \left(\log \hat{x} - \frac{\sigma^2}{2} t^{2H} \right), \quad (10)$$

derive

$$\frac{\partial u(\tau, \hat{y})}{\partial \tau} - \frac{1}{2} \frac{\partial^2 u(\tau, \hat{y})}{\partial \hat{y}^2} = 0. \quad (11)$$

This is the diffusion equation. The solution of this equation is given as,

$$u(\tau, \hat{y}) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{\hat{y}^2}{2\tau}\right). \quad (12)$$

This solution can be interpreted as the kernel for data fluctuating with fractional Brownian motion.

5.2 Metric of bulk in dissipative kernel/metric correspondence

We consider the stochastic deviation from a definite space which follows fractional Brownian motion [6],

$$d\tilde{x}_i = dx_i + d\hat{x}_i. \quad (13)$$

The metric with this deviation becomes,

$$\begin{aligned} d\tilde{s}^2 &= ds^2 + g_{ij} \{ \sigma_i \sigma_j dz_i^H dz_j^H + \sigma_i \sigma_j (dx_i dz_j^H + dx_j dz_i^H) \} \\ &= ds^2 + g_{ij} \sigma_i \sigma_j dz_i^H dz_j^H + o(dt^{1+H}) \\ &\sim ds^2 + g_{ij} \sigma_i \sigma_j dt^{2H} \mathbf{e}^i \mathbf{e}^j. \end{aligned}$$

Here, $\mathbf{e}_i = \frac{dx_i}{|dx_i|}$. We take $H = 1 - \epsilon$ and expand for small ϵ ,

$$\begin{aligned} dt^{2H} &= dt^{2(1-\epsilon)} \\ &\sim dt^2 (1 - 2\epsilon \ln(dt)). \end{aligned} \quad (14)$$

This gives,

$$d\tilde{s}^2 \sim g_{\mu\nu} dx^\mu dx^\nu + g_{ij} \sigma_i \sigma_j \{1 - 2\ln(dt)\epsilon\} dt^2 \mathbf{e}^i \mathbf{e}^j. \quad (15)$$

Next, we derive the metric from kernel

$$\hat{g}_{ij}(\hat{x}) = \left. \frac{\partial^2}{\partial \hat{x}_i \partial \hat{x}'_j} u(\tau, \hat{x} - \hat{x}') \right|_{\hat{x}=\hat{x}'} \quad (16)$$

$$= \left. \frac{\partial \hat{y}_i}{\partial \hat{x}_i} \frac{\partial \hat{y}'_j}{\partial \hat{x}'_j} \frac{\partial^2}{\partial \hat{y}_i \partial \hat{y}'_j} u(\tau, \hat{y} - \hat{y}') \right|_{\hat{y}=\hat{y}'} \quad (17)$$

Here,

$$\left. \frac{\partial^2}{\partial \hat{y}_i \partial \hat{y}'_j} u(\tau, \hat{y} - \hat{y}') \right|_{\hat{y}=\hat{y}'} = \frac{1}{\sqrt{\pi}(2\tau)^{3/2}} \quad (18)$$

$$= \frac{1}{\sqrt{2^3 \pi t^{3H}}} \quad (19)$$

and

$$\frac{\partial \hat{y}_i}{\partial \hat{x}_i} = \frac{1}{\sigma_i \hat{x}_i}. \quad (20)$$

these give the metric,

$$\hat{g}_{ij}(\hat{x}) \sigma_i \sigma_j = \frac{1}{\hat{x}_i \hat{x}'_j \sqrt{2^3 \pi t^{3H}}}. \quad (21)$$

Notice that the the order of coordinates of this equation is the same as the metric of AdS space.

If $\hat{x}_i \sim e^{-\frac{\sigma_i^2}{2} t^{2H}}$, the metric will be

$$\hat{g}_{ij}(\hat{x}) \sigma_i \sigma_j \sim \frac{e^{\frac{\sigma_i^2 + \sigma_j^2}{2} t^{2H}}}{\sqrt{2^3 \pi t^{3H}}} \quad (22)$$

$$\sim \frac{e^{at^2}}{\sqrt{2^3 \pi t^3}} \left\{ 1 - 2 \left(at^2 - \frac{3}{2} \right) \ln(t) \epsilon \right\}, \quad (23)$$

here, $a = \frac{\sigma_i^2 + \sigma_j^2}{2}$. The comparison of eq. (15) and (23) gives

$$t^2 = \frac{5}{\sigma_i^2 + \sigma_j^2}. \quad (24)$$

This gives the scale of time to which kernel/metric correspondence holds indifferent to Harst constant.

6 Comments on causality

The time-correlation between past and future is calculated as follows [7]:

$$\begin{aligned}
C_{\Delta z^H}(t) &= \frac{\mathbb{E}[(z^H(t) - z^H(t - \Delta t))(z^H(t + \Delta t) - z^H(t))]}{\mathbb{E}[(z^H(t) - z^H(t - \Delta t))^2]} \\
&= \frac{\sigma^2(2^{2H-1} - 1)|\Delta t|^{2H}}{\sigma^2|\Delta t|^{2H}} \\
&= 2^{2H-1} - 1
\end{aligned}$$

This derived correlation becomes positive for $1/2 < H < 1$, zero for $H = 1/2$, and negative for $0 \leq H < 1/2$.

The analysis of dimension to $d\hat{x} = \sigma\hat{x}dz^H$ shows that physical causality is broken for $H < 1$, because the information travels in proportion to the time of order of lower than 1. If the broken causality is very small, the analysis with small ϵ of the last section will be effective.

7 Conclusions

In this paper, we studied the kernel/metric correspondence in information theory, when accompanying fractional Brownian motion. This corresponds to a special case of our previous work. However, this is merely one of reconfirmations of Amari's great works.

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